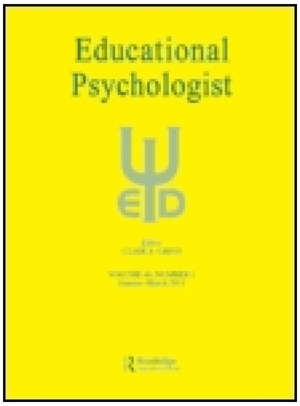

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Teaching the Conceptual Structure of Mathematics

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Teaching the Conceptual Structure of Mathematics

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Many students graduate from K–12 mathematics programs without flexible, conceptual mathematics knowledge. This article reviews psychological and educational research to propose that refining K–12 classroom instruction such that students draw connections through relational comparisons may enhance their long-term ability to transfer and engage with mathematics as a meaningful system. We begin by examining the mathematical knowledge of students in one community college, reviewing results that show even after completing a K–12 required mathematics sequence, these students were unlikely to flexibly reason about mathematics. Rather than drawing relationships between presented problems or inferences about the representations, students preferred to attempt previously memorized (often incorrect) procedures (Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010). We next describe the relations between the cognition of flexible, comparative reasoning and experimentally derived strategies for supporting students' ability to make these connections. A cross-cultural study found that U.S. teachers currently use these strategies much less frequently than their international counterparts (Hiebert et al., 2003; Richland, Zur, & Holyoak, 2007), suggesting that these practices may be correlated with high student performance. Finally, we articulate a research agenda for improving and studying pedagogical practices for fostering students' relational thinking about mathematics.

Many schools are failing to teach their students the conceptual basis for understanding mathematics that could support flexible transfer and generalization. Nowhere is this lack of a conceptual base for mathematical knowledge more apparent than among the population of American students who have successfully graduated from high school and entered the U.S. community college system (Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010). These community college students have completed the full requirements of a K–12 education in the United States and made the motivated choice to seek higher education, but typically without the financial resources or academic scores to enter a 4-year institution. Despite having completed high school successfully, based on entry measures the majority of these students place into "developmental" or "remedial" mathematics courses (e.g., Adelman, 1985; Bailey, Jenkins, & Leinbach, 2005). Too often, these remedial courses then turn into barriers that impede progress toward a higher level degree (Bailey, 2009).

The numbers of community college students in the United States who cannot perform adequately on basic mathematics assessments provide some insight into the questionable efficacy of the U.S. school system. More broadly, detailed measures of these students' knowledge further elucidate the ways in which K–12 educational systems (in any country) have the potential to misdirect the mathematical thinking of many students. We begin this article by describing the results of detailed assessment and interview data from students in a California community college to better understand some longer term outcomes of a well-studied K–12 educational system (Givvin et al., 2011; Stigler et al., 2010). To anticipate, the mathematics knowledge of these students appears

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to be largely bound to specific procedures, leaving the students ineffective at reasoning through a mathematics problem. They are apt to attempt procedures that are partially or incorrectly recalled without regard to the reasonableness of the solution.

We then consider what may be missing from typical U.S. K–12 mathematics instruction, a gap that leads to such impoverished knowledge representations. In particular, we consider one key to developing flexible and conceptual understanding: comparing representations and drawing connections between them. This topic has been the focus of a great deal of cognitive and educational research, enabling us to forge relationships between these literatures to draw implications for pedagogical practice. An integration of these literatures leads us to posit the crucial roles of developing causal structure in knowledge representations, and in supporting students in learning to represent novel problems as *goal-oriented structured systems*.

The term "conceptual understanding" has been given many meanings, which in turn has contributed to difficulty in changing teacher practices (e.g., see Skemp, 1976). For our purposes in this article, we rely on a framework proposed by Hatano and Inagaki (1986), which characterizes conceptual understanding as attainment of an expertlike fluency with the conceptual structure of a domain. This level of understanding allows learners to think generatively within that content area, enabling them to select appropriate procedures for each step when solving new problems, make predictions about the structure of solutions, and construct new understandings and problem-solving strategies. For the sake of clarity, rather than discussing "conceptual understanding" throughout his article, we primarily focus our review of the literature and research agenda on the goal of facilitating learners' acquisition of the *conceptual structure* of mathematics.

We next turn from consideration of student knowledge to studies of videotaped teacher practice, to examine the alignment between current teacher practice and the strategies we hypothesize to be effective. We find that the practices of American teachers often do not correspond at all well with the strategies that we believe would promote deep learning and acquisition of the conceptual structure of mathematics. Finally, we consider the role that researchers can play in understanding how teachers might practicably engage students in effective representational thinking. We lay out a research agenda with the aim of developing strategies for facilitating students' learning to reason about mathematics and to generalize their mathematical knowledge.

WHAT COMMUNITY COLLEGE STUDENTS KNOW ABOUT MATHEMATICS

Studying the U.S. mathematics instructional system provides insights into more general relationships between student knowledge, student cognition, and teacher practices. We know from international research that American students fall far behind their counterparts in other industrialized nations, both on standardized tests of mathematics achievement (Gonzales et al., 2008) and on tests designed to measure students' abilities to apply their knowledge to solving novel and challenging problems (Fleishman, Hopstock, Pelczar, & Shelley, 2010). We also know that the gap between U.S. students and those in other countries grows wider as students progress through school, from elementary school through graduation from high school (Gonzales et al., 2008).

Many researchers have attributed this low performance in large part to the mainly procedural nature of the instruction American students are exposed to in school (e.g., Stigler & Hiebert, 1999). By asking students to remember procedures but not to understand when or why to use them or link them to core mathematical concepts, we may be leading our students away from the ability to use mathematics in future careers. Perhaps nowhere are the results of our K–12 education system more visible than in community colleges. As previously noted, the vast majority of students entering community college are not prepared to enroll in a college-level mathematics class (Bailey, 2009). We know this, mostly, from their performance on placement tests. But placement tests provide only a specific type of information: They measure students' ability to apply procedural skills to solving routine problems but provide little insight into what students actually understand about fundamental mathematics concepts or the degree to which their procedural skills are connected to understandings of mathematics concepts.

American community college students are interesting because they provide a window for examining long-term consequences of a well-studied K–12 instructional system. Not everyone goes to community college, of course. Some students do not continue their education beyond the secondary level, and some American students, through some combination of good teaching, natural intelligence, and diligent study, learn mathematics well in high school and directly enter 4-year colleges. Some community college students pass the placement tests and go on to 4-year colleges, and some even become mathematicians. However, we believe much can be learned from examining the mathematical knowledge of that majority of community college students who place into developmental mathematics courses. Most of these students graduated high school. They were able to remember mathematical procedures well enough to pass the tests in middle school and high school. But after they stop taking mathematics in school, we can see what happens to their knowledge—how it degrades over time, or perhaps was never fully acquired in the first place. The level of usable knowledge available to community college students may tell us something about the long-term impact of the kinds of instructional experiences they were offered in their prior schooling.

We begin by looking more closely at what developmental mathematics students in community college know and understand about mathematics. Little is known about the mathematical knowledge of these students. Most of what we know was learned in two small studies—one a survey of several hundred students at one Los Angeles area college (Stigler et al., 2010), and the other a more qualitative interview study in which interviewers engaged students in conversations about mathematics (Givvin et al., 2011). Both of these studies steered clear of the typical, procedural questions asked on placement tests. Students were not asked to multiply fractions, perform long division, or solve algebraic equations. The questions focused instead on very basic concepts: Could students, for example, place a proper fraction on a number line, or use algebraic notation (e.g., $a + b = c$) to reason about quantitative relationships? More general questions were also asked, such as, what does it mean to do mathematics? We briefly summarize some of the conclusions from these two studies.

Students View Mathematics as a Collection of Rules and Procedures to Be Remembered

Consistent with the view that K–12 mathematics instruction focuses primarily on practicing procedures, these students for the most part have come to believe that mathematics is not a body of knowledge that makes sense and can be "figured out." Instead, they see mathematics as a collection of rules, procedures, and facts that must be remembered—a task that gets increasingly more difficult as students progress through the curriculum.

When asked what it means to be "good at mathematics," 77% of students presented views consistent with these beliefs (Givvin et al., 2011). Here is a sampling of what they said:

- "Math is just all these steps."
- "In math, sometimes you have to just accept that that's the way it is and there's no reason behind it."
- "I don't think [being good at math] has anything to do with reasoning. It's all memorization."

This is, of course, a dysfunctional view of what it means to do mathematics. If students don't believe that it is possible to reason through a mathematics problem, then they are unlikely to try. And if they don't try to reason, to connect problems with concepts and procedures, then it is hard to imagine how they would get very far in mathematics.

Mathematicians, naturally, see reasoning about relationships as central to the mathematical enterprise (e.g., Hilbert, 1900; Polya, 1954), a view that also is common among mathematics teachers at community colleges. When data on students' views of mathematics were presented to a community college mathematics department, the faculty members were astounded. One said, "The main reason I majored in mathematics was because I didn't have to memorize it, it could all be figured out. I think I was too lazy to go into a field where you had to remember everything." Every one of the other faculty members present immediately voiced their agreement.

Given this disconnect between the students and their community college professors, one might ask where the students' views of mathematics come from, if not from their teachers? First, it is important to point out that they bring this view with them based on their K–12 experiences. But it also is quite possible that students' views of what it means to do mathematics arise not from the beliefs of their teachers but from the daily routines that define the practice of school mathematics (see, e.g., Stigler & Hiebert, 1999). Unless teachers' beliefs are somehow instantiated in daily instructional routines or made explicit in some other way, they are unlikely to be communicated to students.

As we see later, the routines of K–12 school mathematics emphasize repeated recall and performance of routine facts and procedures, and these routines are supported by state standards, assessments, and textbooks in addition to teaching practices. Although a small percentage of students do seek meaning and do achieve an understanding that is grounded in the conceptual structure of mathematics—and we assume that community college mathematics faculty are among this small percentage—the majority of students appear to exit high school with a more limited view of what it means to do mathematics.

Regardless of Placement, Students Are Lacking Fundamental Concepts That Would Be Required to Reason About Mathematics

Although the developmental mathematics students in the studies were placed into three different levels of mathematics courses—basic arithmetic, pre-algebra, or beginning algebra—they differed very little in their understanding of fundamental mathematics concepts. Their similarity may not be that surprising given their procedural view of mathematics: If mathematics is not supposed to make sense, consisting mainly of rules and procedures that must be memorized, then basic concepts may not be perceived as useful. That said, the range of things these students did not understand is surprising.

One student, in the interviews, was asked to place the fraction 4/5 on a number line. He carefully marked off a line, labeled the marks from 0 to 8, and then put 4/5 between 4 and 5. Many students appeared to have fundamental misunderstandings of fractions and decimals, not seeing them as numbers that could be compared and ordered with whole numbers. In the survey, students were shown the number line depicted in Figure 1, which spanned a range from –2 to +2. They were asked to place the numbers -0.7 and 13/8 on the number line. Only 21% of the students could do so successfully.

Most young children know that if you add two quantities together to get a third, the third quantity is then composed of the original two quantities such that if you removed one you would be left with the other. The students in these studies, however, seemed happy to carry out their mathematics work without connecting it to such basic ideas. In the interviews

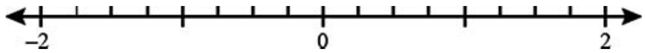

FIGURE 1 Number line on which students were to place −.7 and 13/8. From "What Community College Developmental Mathematics Students Understand About Mathematics," by J. W. Stigler, K. B. Givvin, and B. Thompson, 2010, *The MathAMATYC Educator*, *10*, p. 12. Copyright 2010 by American Mathematical Association of Two-Year Colleges. Reprinted with permission.

a student was asked if he could think of a way to check that the sum of two three-digit numbers was correct. The specific example presented was $462 + 253 = 715$. The student proceeded to subtract 253 from 715 and ended up with 462. The interviewer then asked him if he could have subtracted the 462 instead. He did not think so; he had been told, he said, that you subtract the second addend, not the first. But would it make a difference, the interviewer asked? He wasn't sure. The interviewer told him he could try it, and he did. He seemed genuinely surprised to find that, indeed, he could subtract either addend to get the other.

In one final example, students were asked, "Which is greater? a/5 or a/8. Only 53% correctly answered a/5, a percentage that could have been obtained just by guessing. The students were also asked to explain how they got their answer. About one third (36%) could not come up with an explanation (half of these had answered correctly, the other half incorrectly). The ones who provided some sort of explanation tended to summon some rule or procedure from memory that they thought might do the trick. Many of the students said that a/8 is larger because 8 is larger than 5. Not surprisingly, another group claimed just the opposite, having remembered that the lower the denominator the larger the number. Some students tried to perform a procedure: Some found common denominators, though often they made mistakes and got the wrong answer anyway. Others cross-multiplied (something they apparently believed you can do whenever you have two fractions).

Only 15% of the students tried to reason it through. These students said things such as, "If you have the same quantity and divide it into five parts, then the parts would be larger than if you divide it into eight parts. Assuming you have the same number of these different-sized parts, then a/5 must be larger." Although it is discouraging that only 15% took this approach, it is interesting to note that every one of these students got the answer correct. If we could only figure out how to connect such fundamental ideas with the mathematics procedures students are learning in school, the mathematical knowledge the students acquire might be more robust.

Students Almost Always Apply Standard Procedures, Regardless of Whether They Make Sense or Are Necessary

Students were asked a number of questions in the interviews that could have been answered just by thinking. As evident in the preceding example, only a small percentage of students tried to think their way to a solution. For some questions, just a bit of thinking and reflection might have guided students to use a more appropriate procedure, or to spot errors in the procedures they did use. Rarely, however, did students take the bait.

In one part of the interview students were presented with a list of multiplication problems and asked to solve them mentally:

$$10 \times 3 =$$
$$10 \times 13 =$$
$$20 \times 13 =$$
$$30 \times 13 =$$
$$31 \times 13 =$$
$$29 \times 13 =$$
$$22 \times 13 =$$

Clearly there are many relationships across these problems, and results of previous problems could potentially be used to derive the answers to subsequent problems. But this was not the way in which students approached this task. Most students just chugged through the list, struggling to apply the standard multiplication algorithm to each problem. Fully 77% of the students never noticed or used any relation among the different problems, preferring to work each problem independently.

Here is an example of the answers produced by one student (Givvin et al., 2011):

$$10 \times 3 = 30$$
$$10 \times 13 = 130$$
$$20 \times 13 = 86$$
$$30 \times 13 = 120$$
$$31 \times 13 = 123$$
$$29 \times 13 = 116$$
$$22 \times 13 = 92$$

In summary, most students answered most problems by retrieving answers or procedures from memory. Many of the procedures they used were not necessary or not appropriate to the problem at hand. Rarely were the procedures linked to concepts, which might have guided their use in more appropriate ways. When students were asked to solve multiple problems, they almost never made comparisons across the problems, leading to more mistakes and fewer opportunities to infer the principles and concepts that could make their knowledge more stable, coherent, flexible, and usable.

Why might students have developed such an orientation toward mathematics through their K–12 mathematics education? Although teaching in the United States is multifaceted and the reasons behind student success or failure are much too complex to fully treat here, we consider in particular one candidate explanation: Students do not view mathematics as a *system* because their teachers do not capitalize on opportunities to draw connections between mathematical representations. In the following sections we first expand on what kinds of processes might be required for the development of deep and flexible mathematical knowledge. We then consider, based on classroom observational studies, whether American students have opportunities to engage in these processes. We first consider the cognition involved in students' comparative thinking and transfer, and then we turn to studies of teacher practices to examine alignment between pedagogy and cognition.

LEARNING RELATIONAL STRUCTURE THROUGH COMPARISON

It seems a safe conjecture that the very same students who apparently found no interesting patterns within a series of juxtaposed multiplications by the age of 13 are quite capable of noticing other sorts of potential comparisons and learning from them. They might compare the plots of movies, the sources of difficulty in different video games, the reasons why various romantic relationships have succeeded or failed. In such everyday situations people of all ages, including the very young, spontaneously seek explanations for *why* things happen, especially when faced with surprising events (e.g., Legare, Gelman, & Wellman, 2010). The answer to a "why" question inevitably hinges on relational representations, particularly *cause–effect* relations (for a review, see Holyoak & Cheng, 2011), or more generally (and especially in mathematics), *functional* relations that govern whether inferences are justified (Bartha, 2010)

Learning Schemas Via Analogical Reasoning

A causal model is a kind of *schema*, or mental representation of the relational structure that characterizes a class of situations. The acquisition of schemas is closely related to the ability to compare situations and draw *analogies* based in part on corresponding relations. Analogical reasoning is the process of identifying goal-relevant similarities between what is typically a familiar *source* analog and a novel, less understood *target*, and then using the set of correspondences, or *mapping*, between the two analogs to generate plausible inferences about the latter (see Holyoak, 2012, for a review). The source may be a concrete object (e.g., a balance scale), a set of multiple cases (e.g., multiple problems involving balancing equations), or a more abstract schema (e.g., balancing equations in general). The target may be a relatively similar problem context (e.g., a balancing equations problem with additional steps), or a more remote analog (e.g., solving a proportion).

It has been argued that analogical reasoning is at the core of what is unique about human intelligence (Penn, Holyoak, & Povinelli, 2008). The rudiments of analogical reasoning with causal relations appear in infancy (Chen, Sanchez, & Campbell, 1997), and children's analogical ability becomes more sophisticated over the course of cognitive development (Brown, Kane, & Echols, 1986; Holyoak, Junn, & Billman, 1984; Richland, Morrison, & Holyoak, 2006). Whereas very young children focus on global similarities of objects, older children attend to specific dimensions of variation (Smith, 1989) and to relations between objects (Gentner & Rattermann, 1991).

Analogical reasoning is closely related to transfer. Crucially, comparison of multiple analogs can result not only in transfer of knowledge from a specific source analog to a target (Gick & Holyoak, 1980) but also in the induction of a more general schema that can in turn facilitate subsequent transfer to additional cases (Gick & Holyoak, 1983). People often form schemas simply as a side effect of applying one solved source problem to an unsolved target problem (Novick & Holyoak, 1991; Ross & Kennedy, 1990). The induction of such schemas has been demonstrated both in adults and in young children (e.g., Brown et al., 1986; Chen & Daehler, 1989; Holyoak et al., 1984; Loewenstein & Gentner, 2001). Comparison may play a key role in children's learning of basic relations (e.g., comparative adjectives such as "bigger than") from nonrelational inputs (Doumas, Hummel, & Sandhofer, 2008), and in language learning more generally (Gentner & Namy, 2006). Although two examples can suffice to establish a useful schema, people are able to incrementally develop increasingly abstract schemas as additional examples are provided (Brown et al., 1986; Brown, Kane, & Long, 1989; Catrambone & Holyoak, 1989).

Why Schema Learning Can Be Hard (Especially in Mathematics)

If humans have a propensity to use analogical reasoning to compare situations and induce more general schemas, why did the community college students described earlier appear not to have acquired flexible schemas for mathematical concepts? Several issues deserve to be highlighted. As we have emphasized, most everyday thinking focuses on understanding the physical and social environment, for which causal relations are central (Holyoak, Lee, & Lu, 2010; Lee & Holyoak, 2008). Not all relational correspondences are viewed as equally important. Rather, correspondences between elements causally related to a reasoning goal are typically considered central (Holyoak, 1985; Spellman & Holyoak, 1996). It seems that the human ability to learn from analogical comparisons is closely linked to our tendency to focus on cause–effect relations, which are the building blocks of causal models.

But by its very nature, mathematics is a formal system, within which the key relations are not "causal" in any straightforward sense. (Note that this observation applies not only to mathematics but to other domains as well. For example, a similar pattern of teaching and learning has been identified in the domain of physics by Jonassen, 2010.) Worse, unless mathematical procedures are given a meaningful interpretation, students may assume (as we have seen) that there are no real "reasons" why the procedures work. In some sense, the community college students we interviewed probably did have a "schema" for multiplication, consisting of roles for multiplicands and a product. However, lacking any meaningful model of what multiplication "means" outside of the procedure itself, the students lacked a reliable basis for finding "interesting" relationships between juxtaposed problems such as "$10 \times 13 =$ " and "$20 \times 13 =$ ".

In contrast, their community college professors clearly viewed mathematics as a *meaningful* system, governed by an interconnected set of relations. Though not "causal" per se, these relations are seen as having *relevance* to mathematical goals (Bartha, 2010). As Bartha argued, the general notion of functional relevance (of which causal relations are a special case) governs inference based on mathematics. Just as causal relations determine the consequences of actions in the physical world, mathematical relations determine the validity of procedures in a formal world. For example, multiplication can be defined as repeated addition, which can be defined in turn as the concatenation of two quantities, and quantities can in turn be decomposed (e.g., the quantity 20 is equal to two quantities of 10). This is the type of relational knowledge required to notice, for example, that the value of 20×13 has a special relationship to the value of 10×13. Similarly, the professors, but not their students, understand that numbers in decimal notation like –0.7 and improper fractions like 13/8, along with integers, can all be placed on a number line because all of them are real numbers, representing quantities along a continuum. One might say, then, that the students and their professors have incommensurate schemas for mathematics, in that only the latter place emphasis on functional relations that serve to explain why various mathematical inferences are valid.

Clearly, simply solving sequences of math problems is no guarantee that the student will end up deeply understanding the conceptual structure of mathematics. Even in nonmathematical domains, simply providing multiple examples does not ensure formation of a useful schema. If two examples are juxtaposed but processed independently, without relational comparison, learning is severely limited (Gentner, Loewenstein, & Thompson, 2003; Loewenstein, Thompson, & Gentner, 2003). Even when comparison is strongly encouraged, some people will fail to focus on the goal-relevant functional relations and subsequently fail on transfer tasks (Gick & Holyoak, 1983). When mathematics problems are embedded in specific contexts, details shared by different contexts are likely to end up attached to the learned procedure, potentially limiting its generality. For example, people tend to view addition as an operation that is used to combine categories at the same level in a semantic hierarchy (e.g., apples and oranges, not apples and baskets; Bassok, Pedigo, & Oskarsson, 2008), because word problems given in textbooks always respect this constraint. At an even more basic level, analogical transfer is ultimately limited by the reasoner's understanding of the source analog (Bartha, 2010; Holyoak et al., 2010). If every solution to a math problem is viewed as "just all these steps" with "no reason behind it," simply comparing multiple examples of problems (that to the student are meaningless) will not suffice to generate a deep schema.

Thus, although relational comparisons can in principle foster induction of flexible mathematical knowledge, many pitfalls loom large. The teacher needs to introduce source analogs that "ground out" formal mathematical operations in domains that provide a clear semantic interpretation (e.g., introducing the number line as a basic model for concepts and operations involving continuous quantities). Moreover, even if a good source analog is provided, relational comparisons tax limited working memory (Halford, 1993; Hummel & Holyoak, 1997, 2003; Waltz, Lau, Grewal, & Holyoak, 2000). In general, any kind of intervention that reduces working-memory demands and helps people focus on goal-relevant relations will aid learning of effective problem schemas and thereby improve subsequent transfer to new problems.

For example, Gick and Holyoak (1983) found that induction of a schema from two disparate analogs was facilitated when each analog included a clear statement of the underlying solution principle. In some circumstances, transfer can also be improved by having the reasoner generate a problem analogous to an initial example (Bernardo, 2001). Other work has shown that abstract diagrams that highlight the basic solution principle can aid in schema induction and subsequent transfer (Beveridge & Parkins, 1987; Gick & Holyoak, 1983). Schema induction can also be encouraged by a procedure termed "progressive alignment": providing a series of comparisons ordered "easy to hard," where the early pairs share salient similarities between mapped objects as well as less salient relational correspondences (Kotovsky & Gentner, 1996). More generally, to understand the potential role of analogical reasoning in education, it is essential to consider pedagogical strategies for supporting relational representations and comparative thinking. Next we consider several such pedagogical strategies, including highlighting goal-relevant relations in the source analog, introducing multiple source representations, and using explicit verbal and gestural cues to draw attention to relational commonalities and differences (see also Schwartz, Chase, & Bransford, 2012/this issue; and Chi & VanLehn, 2012/this issue).

How American Teachers Introduce Mathematical Relations

Are teachers invoking these and related strategies in U.S. mathematics instruction, either explicitly or implicitly? Although the list of potential "best practices" in mathematics is long and varied, there is general agreement about the importance of drawing connections and supporting student reasoning. The National Council of Teachers of Mathematics (NCTM) has issued strong recommendations in this vein, publishing a new series of books for high school mathematics under the titled theme of *Reasoning and Sense-Making* (NCTM, 2009). They define "reasoning" broadly, including any circumstance in which logical conclusions are drawn on the basis of evidence or stated assumptions, from informal explanations to deductive and inductive conclusions and formal proofs (p. 19). Sense making is characterized as the interrelated but more informal process of developing understanding of a situation, context, or concept by connecting it with existing knowledge (p. 19). Based on reviews of educational research in mathematics and mathematics education, the authors explore the following theme throughout the main volume in this series as well as in books with specific curriculum foci:

> Reasoning and sense making are the cornerstones of mathematics. Restructuring the high school mathematics program around them enhances students' development of both the content and process knowledge they need to be successful in their continuing study of mathematics and in their lives. (p. 19)

These themes, though under the different title of "Focal Topics in Mathematics," are also central to their description of high quality elementary instruction (NCTM, 2006).

Thus, there is growing consensus in both the psychological and educational research literatures that teaching students effectively requires teaching them to reason with mathematics. Further, there is agreement that this aim necessitates drawing connections and fostering students' awareness that mathematics is a sensible system, one that can be approached using the student's broad repertoire of "sense making," including causal and analogical thinking strategies. Approaching mathematics in this way enables students to develop better structured knowledge representations that may be more easily remembered and used more flexibly in transfer contexts—to solve novel problems, to notice mathematically relevant commonalities and differences between representations, and to reason through mathematics problems when one cannot remember a procedure.

Although drawing connections and sense making do not guarantee transfer, these are cognitive routines that lead to schema acquisition and knowledge representations that support transfer. Positive transfer will be facilitated by noticing similarities between two or more representations or objects.

INTERNATIONAL VARIATIONS IN STUDENTS' OPPORTUNITIES FOR LEARNING TO DRAW CONNECTIONS IN MATHEMATICS

Hiebert and Grouws (2007) conducted a meta-analysis of all studies in which features of teaching were empirically related to measures of students' learning. They found that two broad features of instruction have been shown to promote students' understanding of the conceptual structure of mathematics. First, teachers and students must attend explicitly to concepts, "treating mathematical connections in an explicit and public way" (p. 384). According to Hiebert and Grouws, this could include

> discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas. (p. 384)

The second feature associated with students' understanding of mathematics' conceptual structure is *struggle*: Students must spend part of each lesson struggling to make sense of important mathematics. Hiebert and Grouws defined "struggle" to mean "students expend effort to make sense of mathematics, to figure out something that is not immediately apparent" (p. 387). Thus, students must expend effort to make connections between mathematical problems and the concepts and procedures that can be marshaled to solve them. Note that Hiebert and Grouws did not identify any single strategy for achieving these learning experiences in classrooms, pointing out that there are many ways of doing so. And clearly, not all struggle is good struggle. The point they made is simply that connections must be made *by the student* (i.e., they cannot be made *by* the teacher *for* the student) and the making of these connections will require effort on the student's part.

Corroboration of these conclusions comes from the largest studies ever conducted in which mathematics classrooms have been videotaped in different countries, the TIMSS video studies. Two studies were conducted: the first in 1995 in Germany, Japan, and the United States (Stigler & Hiebert, 1999), and the second in 1999 in seven countries: Australia, the Czech Republic, Hong Kong, Japan, the Netherlands, Switzerland, and the United States (Gonzales et al., 2008; Hiebert et al., 2003). In each country, a national probability sample of approximately 100 teachers was videotaped teaching a single classroom mathematics lesson. An international team of researchers collaboratively developed and reliably coded all lessons to gather data about average teaching practices across and within countries.

One goal of these studies was to try to find features of teaching that might differentiate the high-achieving countries (in general, all except the United States and Australia in the preceding list) from the low-achieving countries. Of interest, the findings fit nicely with the conclusions of Hiebert and Grouws (2007) and also help to explain why we see the kind of outcomes just reported in the studies of community college developmental mathematics students. Many surface features of teaching did not appear associated with cross-national differences in student achievement. For example, among the high-achieving countries, there were countries that emphasized teacher lecture as the primary mode of instruction, and countries that tended to have students work independently or in groups on learning assignments. There were countries that used many real-world problems in their mathematics classes, and countries that dealt almost completely with symbolic mathematics. None of these simple variations could explain differences in student outcomes.

Finding common features among the high-achieving countries required looking more closely at what was happening in the lessons. It was neither the kinds of problems presented nor teaching style employed that differentiated the high-achieving countries from the others, but the kinds of learning opportunities teachers created for students, namely, making explicit connections in the lesson among mathematics procedures, problems, and concepts and finding ways to engage students in the kind of productive struggle that is required to understand these connections in a deep way. The ways that teachers went about creating these learning opportunities differed from country to country. Indeed, an instructional move that inspires a Japanese student to engage might not have the same effect on a Czech student, and vice versa, due to the different motivational beliefs, attitudes, interests, and expectations students in different cultures bring to the task at hand. But the quality of the learning opportunities teachers were able to create did seem to be common across the high-achieving countries.

This conclusion was based on an analysis of the types of problems that were presented, and how they were worked on, in different countries. Across all countries, students spend about 80% of their time in mathematics class working on problems, whether independently, as part of a small group, or as part of the whole class. The beginning and end of each problem was identified as it was presented and worked on in the videos. The types of problems presented were characterized, as was how each was worked on during the lesson.

The two most common types of problems presented were categorized as Using Procedures and Making Connections. Using Procedures problems, by far the most common across all countries, involved asking a student to solve a problem that they already had been taught to solve, applying a procedure they had been taught to perform. This is what is typically regarded as "practice." Take, for example, a lesson to teach students how to calculate the interior angles of a polygon. If the teacher has presented the formula [180 × (number of sides – 2)], and then asks students to apply the formula to calculate the sums of the interior angles of five polygons, that would be coded as Using Procedures. If, however, a teacher asks students to figure out why the formula works, to derive the formula on their own, or to prove that the formula would work for any polygon, that would be coded as a Making Connections problem. A problem like this has the potential for both struggle and for connecting students with explicit mathematical concepts.

The percentage of problems presented in each country that were coded as Using Procedures versus Making Connections is presented in Figure 2. As is evident in the figure, there was great variability across countries in the percentage

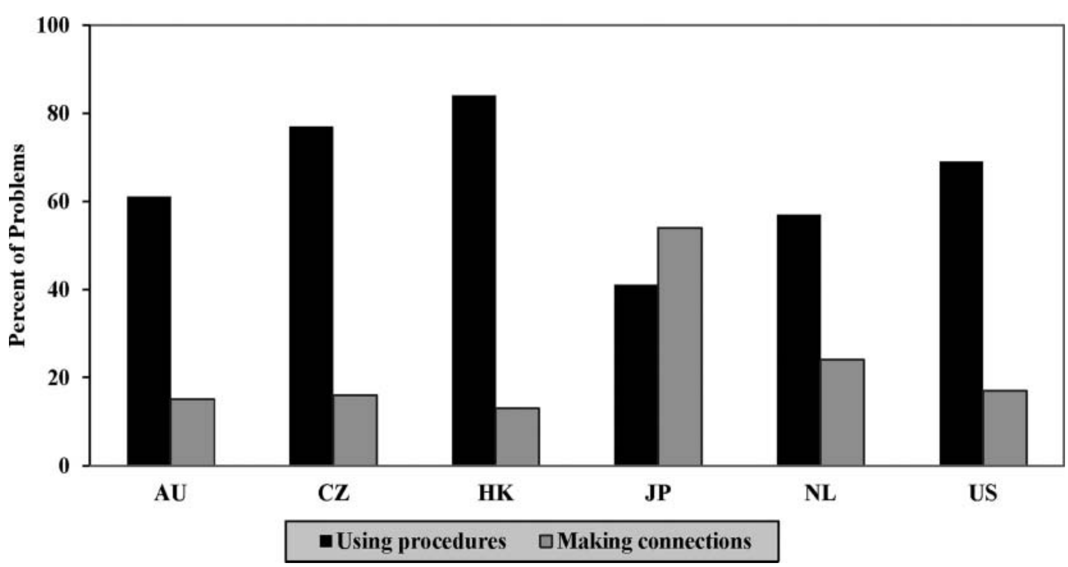

FIGURE 2 Percentage of problems that were coded Using Procedures and Making Connections.

of problems of each type. All countries have some Making Connections problems, though only Japan has more Making Connections problems than Using Procedures problems. Clearly, just presenting more Making Connections problems does not appear to be related to student achievement. Two of the highest achieving countries in the group are Hong Kong and Japan. Hong Kong has the lowest percentage of Making Connections problems, and Japan has the highest. The United States, it is interesting to note, falls in between Hong Kong and Japan. This pattern suggests that curriculum change alone (e.g., increasing the percentage of Making Connections problems in a textbook) will not necessarily result in improved learning.

A more compelling pattern emerges when we examine, in the videos, how the presented problems were actually worked on in the lesson. Although "struggle" per se was not coded, each problem was coded a second time to determine whether the teacher and students engaged with the problem in a way that required them to grapple with concepts or draw connections, or whether the teacher or students changed the activity to reduce the conceptual demand. As evidenced by the data, once a Making Connections problem was presented, it was often changed, by the teacher into something else, most commonly a Using Procedures problem. In other words, just because a problem has the potential to engage students in productive struggle with mathematics concepts, it will not necessarily achieve that potential. For example, a teacher might give additional instruction or a worked example to aid the students in solving the Making Connections problem, which means that the activity becomes only practice for students.

In the United States, one of the reasons that problems do not succeed in engaging students in productive struggle is that the students push back! Teaching is a complex system, and teaching routines are multiply determined. A teacher may ask students why, for example, the equation for finding the sum of the interior angles of a convex polygon works. But students may disengage at this point, knowing that the reasons why will not be on the final exam. Reasons why also may be misaligned with the students' emerging sense of what mathematics is all about: a bunch of procedures to be remembered. Cultural and individual views of the nature of intelligence and learning, specifically as they relate to mathematics, and related processes such as stereotype threat, sense of belonging, and self-efficacy, may undermine students' motivation to engage in persistent effort toward achieving a mathematics learning goal (see, e.g., Blackwell, Trzesniewski, & Dweck, 2007; Dweck & Leggett, 1988; Heine et al., 2001; Walton & Cohen, 2007, 2011).

But teaching practice also may be limited by teachers' own epistemological beliefs about mathematics and how to learn it. Although many K–12 teachers espouse the importance of teaching for "conceptual understanding," the meaning of this phrase has quite variable interpretations (see Skemp, 1976). Because the ability to successfully complete mathematics problems requires both conceptual and procedural skills, teachers regularly find these difficult to distinguish, and may define conceptual understanding and successful learning as comfort with procedures. For this reason, again we find it useful to articulate our hypothesis that students will be best served by learning to represent mathematics as a system of conceptual relationships in which problems and concepts must be connected.

Figure 3 presents just the Making Connections problems, showing the percentage of problems that were actually

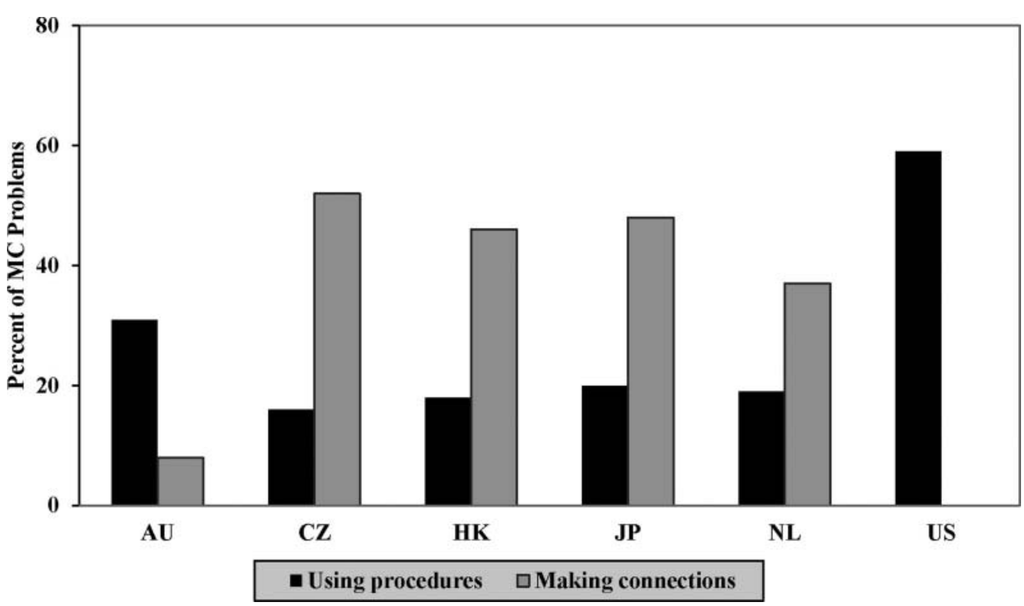

FIGURE 3 Percentage of making-connections problems that were implemented as Using Procedures and Making Connections. Note: Bars do not add up to 100% because there were other transformations that sometimes occurred.

implemented as Making Connections problems versus those that were transformed into Using Procedures problems. Consider Hong Kong and Japan: Whereas they looked completely different when comparing the percentage of problems presented, they look very similar when we look just at how the Making Connections problems are implemented. Both Hong Kong and Japan, and most of the other countries too, are able to realize the full potential of Making Connections problems approximately half the time. The United States, now, is the outlier. Virtually all of the Making Connections problems presented in the United States were transformed into Using Procedures problems, or something requiring even less student conceptual participation. (The reason the percentages do not add up to 100 is that teachers sometimes did other things with Making Connections problems, e.g., just giving the students the answer without allowing them the opportunity to figure it out.)

The kinds of comparison processes that would be required for conceptual learning of mathematics would tend to happen during these Making Connections problems. But for a variety of reasons, such processes do not occur, at least with much frequency, in the U.S. classrooms.

Similar patterns were revealed in smaller scale, more detailed analyses of subsets of the TIMSS video data (Richland, Holyoak, & Stigler, 2004; Richland, Zur, & Holyoak, 2007). Richland et al. (2007) focused specifically on structured analogies, or opportunities for drawing connections and comparative reasoning. These investigators examined a subset of the United States, Hong Kong, and Japanese videotaped lessons to identify teacher practices in using and supporting students in making comparisons between problems, representations, or concepts. These included opportunities for comparisons between problems (e.g., "These are both division problems but notice this one has a remainder") between mathematical concepts (e.g., between convex and concave polygons), between mathematics and nonmathematics contexts (e.g., "an equation is like a balancing scale"), or between multiple student solutions to a single problem.

Every instance identified as a comparison was coded to reveal teachers' strategies for supporting students in drawing the connections intended by the teacher. An international team coded the videos, with native speakers from each country, yielding high reliability across all codes. Because (as previously discussed) the cognitive science literature on comparative reasoning indicates that novices in a domain often fail to notice or engage in transfer and comparative thinking without explicit cues or support, the codes were designed to determine the extent to which teachers were providing such aids. The codes were developed based on the cognitive science literature and on teacher practices observed in other TIMSS videotaped lessons, in an iterative fashion.

Specifically, the codes measured teacher instructional practices that could be expected to encourage learners to draw on prior causal knowledge structures and reduce working memory processing load. The codes assessed the presence or absence of the following teacher practices during comparisons: (a) using source analogs likely to have a familiar causal structure to learners (vs. comparing two new types of problems or concepts), (b) producing a visual representation of a source analog versus only a verbal one (e.g., writing a solution strategy on the board), (c) making a visual representation of the source analog visible during comparison with the target (e.g., leaving the solution to one problem on the board while teaching the second, related problem), (d) spatially aligning written representations of the source and target analogs to highlight structural commonalities (e.g., using spatial organization of two problem solutions on the board to identify related and unrelated problem elements), (e) using gestures that moved comparatively between the source and target analogs, and (f) constructing visual imagery (e.g. drawing while saying "consider a balancing scale").

Teachers in all countries invoked a statistically similar number of relational comparisons (means of 14–20 per lesson). (These are different from the numbers of Making Connections problems identified in the analysis described previously, as these included also additional types of opportunities for drawing relationships.) Of interest, the data revealed that the U.S. teachers were least likely to support their students in reasoning comparatively during these learning opportunities. These findings were highly similar qualitatively to those from the overall TIMSS results, suggesting that U.S. teachers are not currently capitalizing on learning opportunities (i.e., opportunities for comparison) that they regularly evoke within classroom lessons. Both teachers in Hong Kong and Japan used all of the coded support strategies more often than did the U.S. teachers. As shown in Figure 4, some strategies were used frequently, others less often, but the Asian teachers were always more likely to include one or more support strategies with their comparisons than were teachers in the United States.

Overall, these data suggest that although the U.S. teachers are introducing opportunities for their students to draw connections and reason analogically, there is a high likelihood that the students are not taking advantage of these opportunities and are failing to notice or draw the relevant structural connections. At this point, we have come full circle in our discussion and return back to the students with whom we started. Community college developmental mathematics students don't see mathematics as something they can reason their way through. For this reason, and no doubt other reasons as well, they do not expend effort trying to connect the procedures they are taught with the fundamental concepts that could help them understand mathematics as a coherent, meaningful system. The roots of their approach to mathematics can be seen in K–12 classrooms, where, it appears, teachers and students conspire together to create a mathematics practice that focuses mostly on memorizing facts and step-by-step procedures. We know from research in the learning sciences what it takes to create conceptual coherence and flexible knowledge representations that support transfer. But

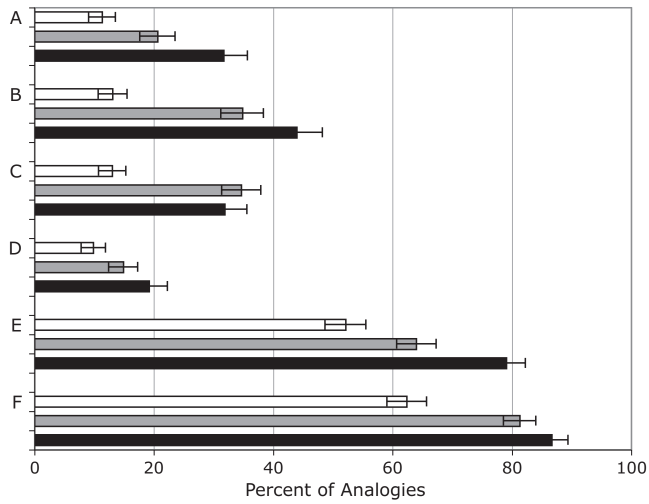

FIGURE 4 Percentage of analogies by region containing cognitive supports: (A) visual and mental imagery, (B) comparative gesture, (C) visual alignment, (D) use of a familiar source, (E) source visible concurrently with target, (F) source presented visually. *Note.* White denotes U.S. teachers, gray denotes Chinese teachers, and black denotes Japanese teachers. From "Cognitive Supports for Analogy in the Mathematics Classroom," by L. E. Richland, O. Zur, and K. J. Holyoak, 2007, *Science*, *316*, p. 1128. Copyright 2007 by the American Association for the Advancement of Science. Reprinted with permission.

we have found it difficult to implement such ideas in classrooms. Where do we go from here?

HOW MIGHT TEACHERS BETTER SUPPORT STUDENTS IN SYSTEMATIC MATHEMATICAL THINKING? A RESEARCH AGENDA

The well-established cognitive literature on learning by structured comparisons, together with our analysis of current teacher practice and student outcomes in mathematics instruction, provides insight into strategies that might better leverage students' reasoning capacities to lead to meaningful understanding of mathematics. We propose several directions for research that would develop a foundation for integrating these ideas into classroom teaching. The first are experimental studies directly investigating the relationship between the pedagogical practices of comparison and analogy and student learning for flexible, transferable mathematics. Second, we outline the importance of understanding teacher and student epistemologies of mathematics in general, and epistemologies of comparison and cognition more specifically. This is essential to understanding the origin of the current problem and to develop recommendations that are likely to have an impact on practice. Third, we call for research on professional development strategies because the problem of how to impact teacher routines, particularly in this area of supporting students' connected, transferable thinking, have proven difficult.

Classroom Efficacy Tests of Strategies for Supporting Comparisons

Although the TIMSS 1999 video data results just reviewed are provocative, they do not allow us to make causal inferences about the relationship between teacher practices and student learning. Several projects have begun to experimentally test the strategies for supporting comparisons that were identified as more frequent in the high-achieving countries (e.g., Richland & McDonough, 2010). So far, this work has found that using a combination of the most common support cues invoked by teachers in Japan and Hong Kong was not necessary to teach basic memorization and use of an instructed strategy, but these cues did increase students' flexibility and ability to identify relevant similarities and differences between instructed problems and transfer problems.

We and other research groups are addressing the question of how to best design and support instructional comparisons. Our team is using controlled videotaped presentation of varied instruction, whereas other research groups are designing tools that aid teachers in leading instruction by comparisons as well as studying comparisons made by peers (see Rittle-Johnson & Star, 2007, 2009; Star & Rittle-Johnson, 2009). More work is needed to investigate strategies for optimizing teachers' current use of problems and comparisons that could be used to encourage students to draw connections and reason meaningfully about mathematics.

Specifically, one of the strategies that needs further research is to better understand how students' prior knowledge structures are related to the types of representations and comparisons that are of most use in supporting sense making and relational reasoning. Adequate prior knowledge is essential for reasoning by comparison, primarily because without awareness of the fundamental elements of a representation, one cannot hope to discern the important structural correspondences and draw inferences on that basis (e.g., Gentner & Rattermann, 1991; Goswami, 2002). Yet surprisingly, using very familiar source analogs was the comparison support strategy identified in the TIMSS video studies that was employed proportionally least frequently by teachers in all countries. As reviewed earlier, the lack of well-structured knowledge about the source will limit students' schema formation and generalization from the target, as they are simultaneously acquiring and reasoning about the causal structure of both the source and target analogs. At minimum, the practice of using unfamiliar source analogs will impose high cognitive demands on the learner, making the additional supports for cognitive load even more important to ensure that students have sufficient resources to grapple with the relationships between the two problems.

Despite the challenges of drawing inferences from a relatively unfamiliar source analog, the literature is not clear as to whether generalization from two less well-known analogs can be as effective as between a known and less well-known analog, assuming the learner has access to optimal supports for causal thinking and sense making. Providing multiple representations certainly can be helpful, even when the domain is fairly novel, through a kind of analogical scaffolding (Gick & Holyoak, 1983).

However, this may vary depending on the background knowledge of the learner. Rittle-Johnson, Star, and Durkin (2009) found that general algebraic knowledge about manipulating equations predicted whether students benefited at all from being taught through comparison between two solution strategies. Those who began instruction with better initial algebra intuitions about procedures for balancing equations (even if the procedures were not executed properly) benefited from this type of comparison, whereas those who were less prepared benefited more from serial instruction about two problems without explicit support for comparison, or from comparisons between two problem types. These students were working in collaborative pairs of peers, so those who began the lesson without adequate knowledge may not have had the level of support necessary to surmount the difficulty of aligning and mapping the representations, but it is not clear what types of supports would have been sufficient.

Kalyuga (2007) proposed an "expertise reversal effect" for the role of cognitive load. This could be interpreted to imply that until students have adequate knowledge, they will benefit from all possible efforts to reduce cognitive load, including reducing the instructional objective to have students encode the structure of a new representation. Once students have more expertise, however, they will gradually be able to handle more cognitive load and may actually benefit from more effortful work to align and map between source and target analogs. Thus, the optimal level and role of teacher supports for relational thinking and sense making may shift over the course of students' learning (cf. Koedinger & Roll, 2012).

Overall, research is necessary to better understand the role of individual differences in prior knowledge and optimal relational learning conditions. Relational thinking and alignment between prior conceptual knowledge and new representations may be a way to characterize an important element of the more general construct "struggle" as described by Hiebert and Grouws (2007). According to this construct, the level of struggle must be attenuated based on students' level of prior knowledge so that the requirement to reason causes struggle, yet the challenge is surmountable.

Although theoretically a very powerful framework for understanding the relationship between student learning needs and instructional content, one can imagine that this level of flexible instruction may be very challenging for teachers. In particular, learning to use such strategies is difficult for novice teachers (Stein, Engle, Smith, & Hughes, 2008), and much more research is needed to better understand teachers' beliefs about comparisons and students' analogical reasoning.

Teacher Knowledge and Professional Development

The instructional strategies we have discussed to this point will be heavily reliant on a teacher who orients to mathematics as a meaningful system and is able to flexibly vary his or her instruction based on diagnosis of students' current knowledge states. There are several parts to this description of a hypothesized ideal teacher that may be important to understand before we can know how to realistically integrate cognitive principles of comparison into classroom instruction.

The first pertains to the structured organization of teacher knowledge and beliefs about the role of connections in mathematics learning. In the community college sample, there was a clear distinction between the professors' and students' *orientations* to mathematics, with the professors viewing mathematics as more of a meaningful system than their students. K–12 mathematics teachers may be more similar to their students in their stored knowledge systems of mathematics, however, appearing more focused on rules (Battista, 1994; Schoenfeld, 1988). Several measures have been designed to assess teacher knowledge about mathematics content and about students' mathematical thinking (e.g., Hill, Ball, & Shilling, 2008; Hill, Schilling, & Ball, 2004; Kersting, Givvin, Sotelo, & Stigler, 2010), yet we need to learn more about teachers' beliefs and knowledge about mathematics as a system, specifically with respect to the roles of multiple representations and drawing connections among content. In particular, it will be important to try to discover where students acquire their belief that mathematics is a series of memorized rules.

International studies suggest that despite variability in teacher expertise in the domain within the United States, there may still be differences in the ways that the mathematical knowledge of American teachers is organized when compared with either mathematics domain experts or with teachers in other counties, particularly with respect to the role of interconnections within the content. Ma (1999) found that the U.S. mathematics teachers had taken more mathematics courses than the average Chinese mathematics teachers in her sample, but the Chinese teachers' representations of mathematics were far more systematic, interconnected, and structurally organized. The U.S. teachers tended to represent the mathematics curriculum as linearly organized, whereas the Chinese teachers' representations of the curriculum more closely resembled a web of connections. Further research internationally as well as within the United States may better reveal teachers' underlying conceptualizations of mathematics, with particular attention to the role of interconnections and meaningful systems of relationships, in much the same way we have gathered information from the community college students (Givvin et al., 2011). Greater understanding of teachers' knowledge of mathematics may aid in developing procedures or tools to better facilitate teacher practices and to optimize the effectiveness of comparison strategies as pedagogical tools.

Finally, we join the NCTM and other mathematics teacher educators in calling for further research on professional development strategies for promoting a conceptual shift for teachers from teaching mathematics as memorization of procedures to a structured system of goal-oriented problem solving. We in particular emphasize the need for professional development to support teachers in learning how to represent problems as goal-oriented systems that can be connected meaningfully to other problems, representations, and concepts. As we have identified in the TIMSS analyses, U.S. teachers are not currently leveraging opportunities for drawing connections and thereby encouraging students to organize their knowledge around mathematical relationships. We require research to better understand how to provide such knowledge to teachers in a way that is usable. In addition, it may prove useful to support teachers through better textbooks and resource tools that include more connected, comparison-based suggested instruction.

In sum, we posit that leveraging students' reasoning skills during K–12 mathematics instruction may be a crucial way to enhance their ability to develop usable, flexible mathematics knowledge that can transfer to out-of-school environments. U.S. teachers are not currently providing most students with opportunities to develop meaningful knowledge structures for mathematics, as revealed by studies of community college students' mathematical skills and video-based observations of teacher practices. Cognitive scientific research on children's causal and relational thinking skills provides insights into strategies for supporting students in gaining more sensible, meaning-driven representations of mathematics. However, more research is necessary to determine how these ideas may become effectively integrated into classroom teaching.

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